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# Relic Abundances and the Boltzmann Equation

Mark Srednicki*
*Department of Physics, University of California, Santa Barbara, CA 93106*

## Abstract

I discuss the validity of the quantum Boltzmann equation for the calculation of WIMP relic densities.

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*E–mail: mark@vulcan.physics.ucsb.edu

The quantum Boltzmann equation (QBE) is the starting point for the calculation of the relic density of stable, weakly interacting massive particles (WIMPs) that were once in thermal equilibrium. This describes a number of dark matter candidates (such as the lightest supersymmetric particle, or LSP). A linearized form of the QBE is also used to calculate the power spectrum of fluctuations in the cosmic microwave background. We rely on the results of these calculations, and expect them to be accurate to a high degree of precision.

It is therefore somewhat disconcerting to realize that the theoretical derivation of the QBE is not at all straightforward, and that there can be important corrections. (For a good general discussion of the key issues in the simpler context of nonrelativistic quantum mechanics, see [1].) Furthermore, I am unaware of any experimental verification of the QBE in which the relevant cross sections have been calculated from first principles, with no phenomenological input. Thus astrophysics leads us to a new frontier for the QBE.

In a series of papers, Matsumoto and Yoshimura (MY) [2] have carefully reconsidered the derivation of the QBE for WIMP relic densities. They found a surprising result: a previously unsuspected correction term that becomes dominant at low temperatures, and changes the standard results. Furthermore, this correction can be computed and understood using the methods of thermal field theory; the subtleties in the derivation of the QBE do not enter into it.

Simply stated, the result of MY is that the number density $n$ of a gas of WIMPS, each with mass $M$, in thermal equilibrium at a temperature $T \ll M$ with a gas of massless particles, is given by

$$n = (2\pi)^{-3/2}(MT)^{3/2}e^{-M/T} + c\lambda^2 T^6/M^3 + \dots\ , \tag{1}$$

where $\lambda$ is a coupling constant and $c$ is a numerical factor. The first term represents the usual result for noninteracting particles, and carries a Boltzmann suppression factor $e^{-M/T}$; the second term is a loop correction, and it does not have this suppression. Therefore, no matter how small $\lambda$ is, at sufficiently low temperatures the second term will dominate.

Is this result correct? We must first ask for the definition of $n$. MY use a model of a heavy spin-zero boson (represented by a real scalar field $\varphi$) interacting with a massless spin-zero boson (represented by a real scalar filed $\chi$). The scalar potential is

$$V(\varphi,\chi) = \tfrac{1}{2}M^2\varphi^2 + \tfrac{1}{24}\lambda_\varphi\varphi^4 + \tfrac{1}{24}\lambda_\chi\chi^4 + \tfrac{1}{4}\lambda\varphi^2\chi^2. \tag{2}$$

We assume that $\lambda_\varphi \ll \lambda^2$, $\lambda \ll \lambda_\chi$, and $\lambda_\chi < 1$. This hierarchy among the couplings allows the light $\chi$ particles to function as an efficient heat bath for the heavy $\varphi$ particles. Then, for $T \ll M$, MY define the number density $n$ of heavy particles via

$$n = \frac{1}{M}\langle\mathcal{H}_\varphi\rangle = \frac{1}{M}\left[\frac{\mathrm{Tr}\,\mathcal{H}_\varphi e^{-H/T}}{\mathrm{Tr}\,e^{-H/T}} - \langle 0|\mathcal{H}_\varphi|0\rangle\right], \tag{3}$$

where $H$ is the total hamiltonian, and

$$\mathcal{H}_\varphi = \tfrac{1}{2}\dot\varphi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}M^2\varphi^2 + \text{counterterms} \tag{4}$$

is the free-field part of the $\varphi$ hamiltonian, plus counterterms (some of which involve the $\chi$ field) that are necessary to remove infinities in this composite operator.

Eq. (3) is a highly plausible definition of the heavy particle number density (at low temperatures). However, it does not correspond in any obvious way to how the this number density would be determined experimentally. Standard methods all involve a search for individual, on-shell $\varphi$ particles. Real-world examples of this include present-day dark matter searches, and measurements of the cosmic microwave background radiation. Therefore, it is possible to suspect that the definition (3) is not appropriate, and this is the reason for the surprising lack of Boltzmann suppression in the loop corrections to $n$.

In [3], this issue was investigated using a simple Caldeira–Leggett model [4]. In this exactly solvable model, a "system" (represented by a harmonic oscillator) is coupled to an "environment" (represented by more oscillators) via an "interaction". We found that, at low temperatures, the "interaction" energy was always comparable to the "system" energy, making the identification of the "system" problematic.

Consider now a slight variation of the MY model, in which the heavy field is complex, and carries a conserved U(1) charge, while the light field remains real and neutral [5]. In this model, we can study the charge fluctuations in a given volume $V$. For noninteracting particles, Poisson statistics for the number of positively and negatively charged particles in $V$ results in

$$\langle Q^2 \rangle = nV. \tag{5}$$

We see that the charge fluctuations give us a measurement of the total number of heavy particles in a given volume. If $Q$ represents electric charge, we can in principle measure $\langle Q^2 \rangle$ without tracking individual heavy particles. Furthermore, it seems highly unlikely that weak interactions could significantly modify eq. (5). If $\langle Q^2 \rangle$ is either much larger or much smaller than $nV$, then the movements of positive and negative particles into and out of $V$ would have to be highly correlated (in order to suppress or enhance the charge fluctuations in $V$). This is inconsistent with the usual notion of a gas of particles that move freely and independently between occasional scatterings, and would appear to require strong interactions.

We therefore *define* the number density of heavy particles, for $T \ll M$, via eq. (5). We can now compute the $O(\lambda^2)$ loop corrections to $\langle Q^2 \rangle$, and see whether or not they are Boltzmann suppressed. The answer is: they are. Details of the loop graph evaluation are given in [5], but there is a simple argument for the Boltzmann suppression of $\langle Q^2 \rangle$ to all orders in perturbation theory. Consider the partition function

$$Z = \operatorname{Tr} e^{-(H-\mu Q)/T} \tag{6}$$

where $\mu$ is a chemical potential for the charge $Q$. We then have

$$\langle Q^2 \rangle = T^2 \frac{\partial^2}{\partial \mu^2} \ln Z \bigg|_{\mu=0}. \tag{7}$$

We will now show that all $\mu$-dependent terms in $Z$ are Boltzmann suppressed. We write

$$Z = \sum_\alpha e^{-(E_\alpha - \mu Q_\alpha)/T}, \tag{8}$$

where the sum is over a basis of energy and charge eigenstates. States that yield a $\mu$-dependent contribution to $Z$ must have $Q \neq 0$. However, in a weakly coupled theory, the lowest-energy state with $Q \neq 0$ is a state consisting of a single zero-momentum $\varphi$ particle. This is an exact energy eigenstate with energy $M$; its contribution to $Z$ is obviously Boltzmann suppressed. Since all other states with $Q \neq 0$ have higher energy, their contributions to $Z$ are Boltzmann suppressed as well. Eq. (7) then implies that $\langle Q^2 \rangle$ is Boltzmann suppressed.

In the original model of MY, there is no conserved charge whose fluctuations can be measured to determine $n$. However, is seems likely that any definition of $n$ that corresponds to an experimental measurement will yield a result that is Boltzmann suppressed. Assuming this, a simple mathematical definition of $n$ is $n = (1/M)\rho_{\rm BS}$, where $\rho_{\rm BS}$ represents those terms in the total energy density that have a Boltzmann-suppression factor. Any unsuppressed terms in $\rho$ are to be thought of as corrections to the energy density of the gas of light particles. This interpretation is in fact the usual one in quantum electrodynamics [6], where unsuppressed loop corrections to the total energy density $\rho$ (at temperatures well below the electron mass) are attributed to self-interactions among photons that arise after integrating the electron-positron field out of the functional integral.

We therefore conclude that the QBE for WIMP relic densities should not be corrected in the manner proposed by MY. However, we should keep in mind that the theoretical situation is not as clear as we might like it to be, and so be on the lookout for possible further surprises.

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